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Changing dimension and time: two well-founded and practical techniques for path integration in quantum physics

Werner Fischer, Hajo Leschke and Peter Müller
Institut für Theoretische Physik I, Universität Erlangen-Nürnberg, Staudtstrasse 7, W-8520 Erlangen, Federal Republic of Germany

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Abstract. In a reasonably self-contained presentation mathematical rigour is supplied to the important ideas of solving certain non-Gaussian path integrals by changes of dimension and/or path-dependent time transformations. The resulting genuine path-integral calculus neither requires discretization prescriptions nor sophisticated methods from the theory of stochastic differential equations. The power of the calculus is illustrated by two standard quantum-physics applications. First, the calculation of the time-dependent propagator corresponding to a particle on the half-line in a harmonic plus inverse-square potential is shown to be a simple exercise. Second, the first rigorous derivation of the energy-dependent Green function of the one-dimensional Morse system is given.

1. Introduction

The dynamical properties of a quantum mechanical system governed by a standard Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{q})$ can be expressed in terms of the imaginary-time or Euclidean propagator. (Here and in the following we will use units such that the mass of the particle and Planck's constant $\hbar$ are equal to one.) In Feynman's formulation of quantum dynamics [1] this propagator is represented by a sum over histories or path integral

$$\langle q_b|\exp\{-t\hat{H}\}|q_a\rangle = \int_{x(0)=q_a}^{x(t)=q_b} \mathcal{D}x \exp\left\{-\frac{1}{2}\int_0^t \mathrm{d}t' \left(\frac{\mathrm{d}x}{\mathrm{d}t'}\right)^2\right\} \exp\left\{-\int_0^t \mathrm{d}t'\, V(x(t'))\right\}. \quad (1)$$

Let us call (1) the intuitive Feynman–Kac formula. It has been known for a long time that Gaussian path integrals, corresponding to quadratic potentials V, can be calculated explicitly. However, it is only in the last two decades that techniques for the calculation of certain non-Gaussian path integrals have appeared in the literature.

This paper is concerned with two of these techniques, namely *path-dependent time transformations* and *changes of dimension*. While the first technique may be considered as an analogue of changing variables in ordinary (Riemann, Lebesgue) integrals, the second one is comparable to relating ordinary integrals in different dimensions by exploiting a symmetry.

In the literature path-dependent time transformations are approached in two different ways. The intuitive approach initiated by [2] in 1979 'in a very formal way',

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see the preface in [3], but further elaborated by e.g. [3–11], relies on somewhat *ad hoc* discretization prescriptions of (1). Not unexpectedly, this approach runs into severe but artificial difficulties. This is because these authors never use a bona fide measure on the path space, which in case of (1) is the well known Wiener measure, see e.g. [12, 13]. Accordingly, some of them [8, 14, 15] understand their approach partially as a recipe to obtain results which have to be checked by operator theory in case of doubt. Clearly, this has led to some confusion in the physics community. Even in a recent textbook [3] whole chapters are devoted to 'time-slicing corrections' and 'stabilization procedures' invented to avoid a 'path collapse' which in reality does not exist due to the stochastic nature of the paths.

The rigorous approach to path-dependent time transformations in path integration—inspired by [2]—was initiated by [16] and further elaborated by [17–22]. It can be reduced to a theorem of Dambis, Dubins and Schwarz (1965) in the calculus of stochastic differential equations, see e.g. [23]. These works have not received the attention they deserve, in part because many physicists seem to be not sufficiently familiar with the calculus of stochastic differential equations, in part because physicists want to have a genuine path-integral calculus at their disposal. Fortunately, the basic ideas for such a calculus can be extracted from a recent work [20] containing a rigorous path-integral calculation for the hydrogen atom without using sophisticated methods from stochastic calculus.

It is one of the two main goals of the present paper to isolate the key points of these ideas, elaborate on them and extend them to more general situations. This will be accomplished in an elementary way. The resulting path-integral calculus is rigorous in the sense that the missing details (specification of domains of operators etc.) are only of a technical kind and can be supplied with additional effort.

Concerning the change-of-dimension technique a similar distinction between an intuitive and a rigorous approach can be made. The intuitive approach is based on certain discretization prescriptions for radial path integrals [24–26]. The change-of-dimension technique within these prescriptions can be traced back to [25] where centrifugal barriers are treated. It has been further elaborated by e.g. [8, 9, 27]. Problems due to the discretization are discussed in [3, 8, 28].

It is the other main goal of the present paper to develop a rigorous approach to the change-of-dimension technique in radial path integrals, which is based on the Radon–Nikodym derivative of a Bessel measure with respect to another one [23, 29]. This approach was hinted at in [8, 21]. Again, we will give an elementary proof of the key result using neither discretization prescriptions nor stochastic calculus.

The paper is organized as follows. In section 2 we introduce our notation and a measure-theoretic definition of path integrals with respect to a general Markov process. Section 3 introduces Bessel path integrals as generalized radial path integrals and provides the rigorous tool (statement 1) for the change-of-dimension technique. In section 4 the explicit path-integral calculation of the propagator corresponding to the standard example of a harmonic plus inverse-square potential is shown to be an immediate consequence of statement 1. Section 5, which is independent of sections 3 and 4, presents the formalism of path-dependent time transformations. Its main results, statements 2 and 3, constitute the basis of a well-founded and practical calculus for path-dependent time transformations in path integration. In order to illustrate the power of this calculus in typical quantum-physics applications we give, to our knowledge, the first rigorous derivation of the energy-dependent Green function of the Morse system in section 6. This needs techniques from both sections 4 and 5.

Elementary proofs of the statements can be found in appendices 1 and 2. Appendix 3 checks some conditions needed in section 6.

2. Basic notation, Markov processes and path integrals

In the following it is essential that we do not only deal with Wiener path integrals, that is, integrals over paths of the Wiener process (or Brownian motion) in the d-dimensional Euclidean space $\mathbb{R}^d$. Instead, we will also consider integrals over paths $x(t)$ in a *configuration space* $Q(\subseteq \mathbb{R}^d)$ realizing a more general continuous stationary *Markov process*. Such a process is a collection of random variables indexed by positive time $t \in \mathbb{R}_+$ and taking on values in Q. It is characterized by a *transition density* $m_t(q',q)$, which is the probability density for going from position $x(t_0) = q \in Q$ to position $x(t_0 + t) = q' \in Q$ during time t independent of t_0. The transition density satisfies the natural properties of a continuous *Markov semi-group*:

(i) $$m_t(q',q) \geqslant 0$$

(ii) $$\int_Q \mathrm{d}q'\, m_t(q',q) = 1$$

(iii) $$\lim_{t\downarrow 0} m_t(q',q) = \delta(q'-q)$$

(iv) $$\int_Q \mathrm{d}q''\, m_{t'}(q',q'')m_t(q'',q) = m_{t'+t}(q',q)$$

(v) $$\lim_{t\downarrow 0} \frac{1}{t}\int_{|q'-q|>\varepsilon} \mathrm{d}q'\, m_t(q',q) = 0 \qquad \text{for all } \varepsilon > 0 \text{ and } q \in Q.$$

Throughout this paper we will assume that all paths start at time $t = 0$ at some position $q_a \in Q$, that is $x(0) = q_a$.

By Kolmogorov's extension theorem [12, 30] the transition density m_t induces a unique *measure* $\mathrm{d}M$, which due to (v) is concentrated on the set $C(Q,q_a)$ of all *continuous* paths starting at q_a (see e.g. [31]). This path measure represents the joint probability distribution of the Markov process. We stress that all paths in the *path space* $C(Q,q_a)$ are open ended.

Given the measure $\mathrm{d}M$, one can define a *path integral*

$$\int_{C(Q,q_a)} \mathrm{d}M[x]\, F[x] \tag{2}$$

of a numerical functional F on $C(Q,q_a)$, that is, a weighted sum or mean of the values $F[x]$ over all continuous paths $x(t)$ starting at q_a. Two simple and explicit examples are

$$\int_{C(Q,q_a)} \mathrm{d}M[x] = 1 \tag{3}$$

and

$$\int_{C(Q,q_a)} \mathrm{d}M[x]\, \delta(x(t_n)-q_n)\ldots\delta(x(t_1)-q_1) = m_{t_n-t_{n-1}}(q_n,q_{n-1})\ldots m_{t_1}(q_1,q_a) \tag{4}$$

where $0 < t_1 < t_2 < \ldots < t_n$. While the first example is just the normalization of the measure, the second one is a path-integral statement of the *Markov property*.

For a *cylindrical functional* f_n, that is, a functional depending only on the positions of the paths at a finite number n of given times $0 < t_1 < t_2 < \ldots < t_n$, the Markov property immediately implies

$$\int_{C(\mathcal{Q},q_a)} \mathrm{d}M[x]\, f_n(x(t_1),\ldots,x(t_n)) = \int_{\mathcal{Q}} \mathrm{d}q_n \ldots \int_{\mathcal{Q}} \mathrm{d}q_1\, m_{t_n-t_{n-1}}(q_n,q_{n-1})\ldots m_{t_1}(q_1,q_a) f_n(q_1,\ldots,q_n). \tag{5}$$

Since n may be arbitrarily large, the value of the path integral of a fairly general functional F can be approached by suitable limiting constructions. This is the strategy of general integration theory. It is also at the bottom of most path-integral constructions used by physicists [1, 32].

For a given Markov process (with transition density m_t) there exists a unique *generator* or 'free-particle energy' operator $\hat{T}_m$ acting on a suitable class of numerical functions defined on $\mathcal{Q}$. Using Dirac's notation for the integral kernel or the position representation of an operator, the relation between the transition density of the Markov process and its generator can be written as

$$\langle q'|\exp\{-t\hat{T}_m\}|q\rangle = m_t(q',q). \tag{6}$$

This relation may be viewed as the 'free-particle' limit of the generalized *Feynman–Kac formula* [33, chapter 5, theorem 7.6]

$$\langle q_b|\exp\{-t\hat{H}\}|q_a\rangle = \int_{C(\mathcal{Q},q_a)} \mathrm{d}M[x]\, \delta(x(t)-q_b)\, \exp\left\{-\int_0^t \mathrm{d}t'\, V(x(t'))\right\}. \tag{7}$$

Here $V(q)$ is a sufficiently well-behaved real-valued 'potential-energy' function on $\mathcal{Q}$ and $V(\hat{q})$ the corresponding multiplication operator, which, when added to $\hat{T}_m$, gives the 'total-energy' operator or *Hamiltonian* $\hat{H} := \hat{T}_m + V(\hat{q})$.

A derivation of (7) follows from (5) and (6) by a suitable sequence $\{f_n\}$ of cylindrical functionals. In quantum physics the LHS of (7) is usually called the time-dependent Euclidean *propagator* of $\hat{H}$. It is uniquely related to its Laplace transform

$$\langle q_b|(\hat{H}-E)^{-1}|q_a\rangle = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{tE}\, \langle q_b|\exp\left\{-t\hat{H}\right\}|q_a\rangle \tag{8}$$

which is called the resolvent kernel or energy-dependent *Green function* of $\hat{H}$. For nearly all $\hat{H}$ of interest in quantum physics the Green function is analytic in the energy variable E, if E is not in the spectrum of $\hat{H}$ (see [34, theorem VIII.2]). As it stands, the integral in (8) makes sense only for values of E with a real part less than the lowest spectral value of $\hat{H}$. Nevertheless, its analytic continuation to other values (not in the spectrum of $\hat{H}$) coincides with the Green function.

Let us now have a look at the most important example of a Markov process, the *Wiener process* in $\mathcal{Q} = \mathbb{R}^d$. It is characterized by the transition density

$$w_t^{(d)}(q',q) := (2\pi t)^{-d/2} \exp\left\{-\frac{(q'-q)^2}{2t}\right\} \qquad q,q' \in \mathbb{R}^d \tag{9}$$

and its generator is the ordinary 'kinetic-energy' operator $\frac{1}{2}\hat{p}^2$, where the momentum operator $\hat{p} := -\mathrm{i}\,\frac{\partial}{\partial q}$ is the gradient divided by the imaginary unit. The induced measure $\mathrm{d}W^d$ on $C(\mathbb{R}^d, q_a)$ is called d-dimensional Wiener measure and corresponding path integrals are called *Wiener path integrals*. Equation (7) specializes to the ordinary Feynman–Kac formula [12]

$$\langle q_b|\exp\left\{-t\left(\tfrac{1}{2}\hat{p}^2 + V(\hat{q})\right)\right\}|q_a\rangle = \int_{C(\mathbb{R}^d,q_a)} \mathrm{d}W^d[x]\,\delta(x(t)-q_b)\exp\left\{-\int_0^t \mathrm{d}t'\,V(x(t'))\right\}. \tag{10}$$

The relation to the intuitive Feynman–Kac formula (1) is brought to light by identifying the formal expression

$$\mathcal{D}x\,\exp\left\{-\frac{1}{2}\int_0^t \mathrm{d}t'\,\left(\frac{\mathrm{d}x}{\mathrm{d}t'}\right)^2\right\}$$

with $\mathrm{d}W^d[x]$ and noting that the Dirac delta function pins the paths at time t at position q_b. The contributions from the paths at times larger than t give a multiplicative factor 1 because of the Markov property and the fact that the Wiener measure is normalized to unity. Therefore, the path integrals in (10) and (1) give the same result, even though the paths in (10) are open ended and the paths in (1) are considered—at least in the constructions of many physicists [1]—to stop at time t at position q_b. It is *this* conceptual difference that allows for the rigorous implementation of path-dependent time transformations in path integrals with open-ended paths as in (10). See section 5.

3. Bessel processes and radial path integrals

In this section we will consider an important family of Markov processes with the positive Euclidean half-line as its configuration space, that is, $Q = \mathbb{R}_+$. To this end let $\nu \geqslant 0$ be a real number and let I_ν denote the modified Bessel function of the first kind with index ν [35]. We define a transition density on $\mathbb{R}_+$ by

$$b_t^{(\nu)}(q', q) := \frac{q'}{t}\left(\frac{q'}{q}\right)^\nu \exp\left\{-\frac{(q')^2 + q^2}{2t}\right\} I_\nu\left(\frac{q'q}{t}\right) \qquad q,\, q' \in \mathbb{R}_+\,. \tag{11}$$

The reader may check that (11) satisfies all properties (i)–(v) of a Markov semigroup. This transition density characterizes the *Bessel process* with index ν, cf. [23, p 415]. The induced measure $\mathrm{d}B_\nu$ on $C(\mathbb{R}_+, q_a)$ is called a Bessel measure and corresponding path integrals are called *Bessel path integrals*.

There is a relation between the transition densities of Bessel and Wiener processes [36]

$$b_t^{(d/2-1)}(q', q) = (q')^{d-1}\int \mathrm{d}\Omega_{q'}\, w_t^{(d)}(q', q) \tag{12}$$

which is valid for $d \geqslant 2$. Here q (resp. q') is the Euclidean norm $|.|$ of q (resp. q') and $\int \mathrm{d}\Omega_{q'}$ stands for the integration over the unit sphere in $\mathbb{R}^d$ of the variable q'.

The relevance of Bessel processes is revealed by the following: Equations (5) and (12) show that the Wiener path integral of an isotropic functional $F[\|.\|]$ on $C(\mathbb{R}^d, q_a)$ is equal to its Bessel path integral with index $d/2-1$

$$\int_{C(\mathbb{R}^d,q_a)} \mathrm{d}W^d[x]\ F[\|x\|] = \int_{C(\mathbb{R}_+,q_a)} \mathrm{d}B_{d/2-1}[x]\ F[x] \qquad d \geqslant 2. \tag{13}$$

In this sense a Bessel path integral with index $d/2-1$ appears as the *radial path integral* of a d-dimensional Wiener path integral.

An example of an explicitly solvable Bessel path integral is that relating to the harmonic oscillator with frequency $\omega \geqslant 0$

$$\begin{aligned}\int_{C(\mathbb{R}_+,q_a)} \mathrm{d}B_\nu[x]\ \delta(x(t)-q_b)\ \exp\left\{-\frac{\omega^2}{2}\int_0^t \mathrm{d}t'\ x^2(t')\right\} \\ = q_b\left(\frac{q_b}{q_a}\right)^\nu \frac{\omega}{\sinh(\omega t)} \\ \times \exp\left\{-\frac{\omega}{2}\left({q_b}^2+{q_a}^2\right)\coth(\omega t)\right\} I_\nu\left(\frac{\omega q_b q_a}{\sinh(\omega t)}\right).\end{aligned} \tag{14}$$

For $\nu = d/2-1$ the calculation can be done by applying (13) and integrating out all angular dependencies of the well known propagator of the isotropic harmonic oscillator in $\mathbb{R}^d$ [1, 12, 13]. Not surprisingly, the result obtained thereby remains true for general ν. This is shown in [23, p 430 (3.3)]. (Of course, one could also check the appropriate differential equation.)

We finish our introduction to Bessel processes with the following:

Statement 1. *Let $\mu, \nu \geqslant 0$ be real numbers and let $F_t : C(\mathbb{R}_+, q_a) \longrightarrow \mathbb{R}$ be a functional depending only on the paths up to some time $t \geqslant 0$, then*

$$\begin{aligned}\int_{C(\mathbb{R}_+,q_a)} \mathrm{d}B_\nu[x]\ F_t[x] \\ = \int_{C(\mathbb{R}_+,q_a)} \mathrm{d}B_\mu[x]\ \left(\frac{x(t)}{q_a}\right)^{\nu-\mu} \exp\left\{\left(\mu^2-\nu^2\right)\int_0^t \frac{\mathrm{d}t'}{2x^2(t')}\right\}\ F_t[x].\end{aligned} \tag{15}$$

Statement 1 will be shown in appendix 1 in an elementary way for functionals needed in this paper, that is, for *Feynman–Kac functionals*

$$F_t[x] = \delta(x(t)-q_b)\ \exp\left\{-\int_0^t \mathrm{d}t'\ V(x(t'))\right\}. \tag{16}$$

The general result is found in [29] and [23, p 419 (1.22)].

In view of (13) the occurrence of the exponential factor in (15) can be understood physically from the fact that the radial motion of a free particle in $\mathbb{R}^d$ with given angular momentum $\sqrt{l(l+d-2)}$ is the same as that of a free particle in $\mathbb{R}^{d+2l}$ with zero angular momentum. Mathematically speaking, the exponential factor is the Radon–Nikodym derivative of the two Bessel measures involved—the analogue of the Jacobian in ordinary integration theory. In combination with (13) statement 1 gives meaning to intuitive manipulations of radial path integrals (see e.g. [3, 8]). In particular, it is important for applications when one wants to eliminate (or create) inverse-square potentials by changing dimensions, that is, by changing the index of a Bessel process.

4. Application: elimination of inverse-square potentials

We consider a particle on the positive Euclidean half-line $Q = \mathbb{R}_+$ under the influence of the potential $V(q)$ and an additional infinitely high potential barrier at the origin corresponding to a Dirichlet boundary condition. Denoting the standard Hamiltonian $\frac{1}{2}\hat{p}^2 + V(\hat{q})$ subjected to this boundary condition by $\hat{H}_{\mathrm{D}}$, a path-integral representation of the corresponding propagator reads

$$\langle q_b | \exp\left\{-t\hat{H}_{\mathrm{D}}\right\} | q_a \rangle = \frac{q_a}{q_b} \int_{C(\mathbb{R}_+, q_a)} \mathrm{d}B_{1/2}[x]\ \delta(x(t) - q_b)\ \exp\left\{-\int_0^t \mathrm{d}t'\ V(x(t'))\right\}. \quad (17)$$

There are two ways to understand the validity of (17). First, we remark that

$$(q/q')\, b_t^{(1/2)}(q', q) = w_t^{(1)}(q', q) - w_t^{(1)}(-q', q) \quad (18)$$

is the (non-normalized) transition density on the half-line obtained from the one-dimensional Wiener process by the method of images [32]. It embodies the absorbing character of the boundary condition, that is, it ensures that almost all paths do not hit the origin. For the details and more general boundary conditions, see [37].

Second, one can start from (7) with $Q = \mathbb{R}_+$ and $\mathrm{d}M = \mathrm{d}B_{1/2}$. By following the lines in appendix 1, one writes the required generator of the Bessel process with index $\frac{1}{2}$ as $\frac{1}{2}\,\hat{q}\,\hat{p}^2\,\hat{q}^{-1}$ and transforms (7) into (17).

Let us now consider potentials of the form $V(q) = U(q) + g/(2q^2)$, where $U(q)$ is less singular than $1/q^2$ at $q = 0$ and $g \geqslant -\frac{1}{4}$ to prevent the particle from 'falling to the center', see [38, 39]. Then (17) and statement 1 imply for all $\nu \geqslant 0$

$$\langle q_b | \exp\left\{-t\hat{H}_{\mathrm{D}}\right\} | q_a \rangle = \left(\frac{q_a}{q_b}\right)^{\nu+1/2} \int_{C(\mathbb{R}_+, q_a)} \mathrm{d}B_\nu[x]\ \delta(x(t) - q_b) \times \exp\left\{-\int_0^t \mathrm{d}t' \left(U(x(t')) + \frac{g + 1/4 - \nu^2}{2x^2(t')}\right)\right\}. \quad (19)$$

To eliminate the $1/x^2$– term we choose ν such that

$$\nu = \sqrt{g + \tfrac{1}{4}}. \quad (20)$$

Thus it remains to calculate the path integral

$$\int_{C(\mathbb{R}_+, q_a)} \mathrm{d}B_\nu[x]\ \delta(x(t) - q_b)\ \exp\left\{-\int_0^t \mathrm{d}t'\ U(x(t'))\right\} \quad (21)$$

for ν determined by (20).

The simple example $U(q) = \frac{1}{2}\omega^2 q^2$, $\omega \geqslant 0$, is of some interest, in part because the resulting potential $V(q)$ arises from the separation of an exactly solvable three-body problem [40], in part because it arises in the angular momentum expansion of the propagator of the isotropic harmonic oscillator. The corresponding path integral (21) was already given in (14). Hence we obtain the final result

$$\langle q_b| \exp\left\{-\frac{1}{2}t\,(\hat{p}^2+\omega^2\hat{q}^2+g/\hat{q}^2)_{\mathrm{D}}\right\}|q_a\rangle$$
$$= (q_b q_a)^{1/2}\,\frac{\omega}{\sinh(\omega t)}$$
$$\times \exp\left\{-\frac{\omega}{2}\,({q_b}^2+{q_a}^2)\coth(\omega t)\right\}\, I_{\sqrt{g+1/4}}\left(\frac{\omega q_b q_a}{\sinh(\omega t)}\right). \tag{22}$$

This is in agreement with earlier intuitive calculations [3, 8, 25, 27], which, however, identify Bessel functions by their asymptotics or require 'proper time-slicing corrections'.

As an aside we mention that an explicit expression for the Green function of $\frac{1}{2}\,(\hat{p}^2+\omega^2\hat{q}^2+g/\hat{q}^2)_{\mathrm{D}}$ can be obtained by Laplace transforming (22) with the help of [35, formula 6.669.4]. For the final result see e.g. [3].

5. Path-dependent time transformations

With the help of path-dependent time transformations certain path integrals which cannot be evaluated directly are changed into those which are already known. The basic ideas are as follows: In analogy to the change-of-variable formula in ordinary integration theory, one hopes that an unknown path integral

$$\int \mathrm{d}M[x]\;F[x]$$

can be related to a known one by a change of path $x(t)=(Ky)(t)$. However, this so called *space transformation* does not only change the functional $F[x]$ (to $F[Ky]$), but also the measure $\mathrm{d}M$. If it simplifies the functional, it may make the measure more complicated and nothing is gained. This is where path-dependent time transformations come in. The idea is to think of the paths Ky being parametrized by a different time s and set $x=TKy$, where T is a path-dependent *time transformation* which reparametrizes each path $(Ky)(s)$ individually in terms of the original time t. Again, T affects both the functional and the measure. The trick is to find a pair of space and time transformations which simplifies the functional and leaves the measure sufficiently simple.

The analogue of the change-of-variable formula is the path-integral identity

$$\int \mathrm{d}M[x]\;F[x] = \int \mathrm{d}N[y]\;F[TKy] \tag{23}$$

where $\mathrm{d}M$ now appears as the image of another measure $\mathrm{d}N$ under TK. It is *the* central point, because it relates the known to the unknown path integral; an observation due to [20]. We call (23) the *transformation identity*.

Now two questions have to be answered:

- Given the measure dN and the transformations K and T, is there a simple condition on the transition density m_t underlying the measure dM from which one can infer that the transformation identity (23) holds? The answer will be presented in section 5.2.
- How to proceed with (23) in typical quantum-physics applications? Actually, this is not quite as simple as one might hope and we postpone the answer to section 5.3.

Let us first describe the transformations in detail. For a summary of notation see table 1.

Table 1. Summary of notation for path-dependent time transformations

Configuration space	Elements	Path space	Path	Measure	Comment
$\mathcal{R}$	r, r'	$C(\mathcal{R}, r_a)$	$y(s)$	dN	known
$\mathcal{Q}$	q, q'	$C(\mathcal{Q}, q_a)$	$(Ky)(s)$		
$\mathcal{Q}$	q, q'	$C(\mathcal{Q}, q_a)$	$x(t)$	dM	unknown

5.1. Space and time transformation

We consider a point transformation k from one configuration space $\mathcal{R}$ onto another configuration space $\mathcal{Q}$

$$k : \mathcal{R} \longrightarrow \mathcal{Q} \qquad r \mapsto q = k(r). \tag{24}$$

We assume k to fulfil the following condition:

(C1) k is smooth and onto, that is $k(\mathcal{R}) = \mathcal{Q}$.

Clearly, the point transformation k induces a path transformation K from the path space $C(\mathcal{R}, r_a)$, supposed to be equipped with a measure dN, onto the path space $C(\mathcal{Q}, k(r_a))$

$$K : C(\mathcal{R}, r_a) \longrightarrow C(\mathcal{Q}, q_a) \qquad y(s) \mapsto (Ky)(s) \tag{25}$$

according to the definitions

$$(Ky)(s) := k(y(s)) \qquad q_a := (Ky)(0) = k(r_a). \tag{26}$$

In the following we will refer to the path transformation K simply as the *space transformation* (induced by k).

Besides k we consider a non-negative real-valued function τ defined on the configuration space $\mathcal{Q}$. Given this function we assign to each path Ky parametrized by time s a new individual time $t_{Ky}(s)$ depending on the path's history up to time s

$$t_{Ky} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \qquad s \mapsto t = t_{Ky}(s) \tag{27}$$

by defining

$$t_{Ky}(s) := \int_0^s \mathrm{d}s' \, \tau\Big(((Ky)(s')\Big). \tag{28}$$

Since τ is a local function of the paths Ky, some authors speak of a local time transformation. To allow the interpretation as time, τ must be defined in such way that $t_{Ky}(s)$ fulfils two natural properties:

(C2) $\quad t_{Ky}(s) \quad$ is strictly increasing in s

(C3) $\quad \lim_{s\to\infty} t_{Ky}(s) = \infty.$

These two conditions are meant to hold for *almost all* paths y, that is, for all paths except for a set of paths of $\mathrm{d}N$-probability zero. Condition (**C2**) allows to express the time s in terms of the time t, that is, for each path Ky there exists an inverse mapping $s_{Ky}(t)$ with the property $t_{Ky}(s_{Ky}(t)) = t$. Now we can reparametrize each path $(Ky)(s)$ in terms of its individual time $t_{Ky}(s)$. This is a path transformation T from the path space $\mathcal{C}(Q, q_a)$ onto itself, mapping the paths $(Ky)(s)$ to the paths $x(t)$

$$T : \mathcal{C}(Q, q_a) \longrightarrow \mathcal{C}(Q, q_a) \qquad (Ky)(s) \mapsto x(t) := (TKy)(t) \tag{29}$$

defined by

$$(TKy)(t) := (Ky)(s_{Ky}(t)). \tag{30}$$

In the following we will refer to the path transformation T simply as the path-dependent *time transformation* (induced by τ).

Combining both transformations we get the link between the paths of the known and the unknown path integral

$$TK : \mathcal{C}(\mathcal{R}, r_a) \longrightarrow \mathcal{C}(Q, q_a) \qquad y(s) \mapsto x(t) = (TKy)(t). \tag{31}$$

Let us take the hydrogen atom as an example [20]. The space and time transformations may be defined by

$$k : \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \qquad q = k(r) = \begin{pmatrix} r_1^2 - r_2^2 - r_3^2 + r_4^2 \\ 2r_1r_2 - 2r_3r_4 \\ 2r_1r_3 + 2r_2r_4 \end{pmatrix} \tag{32}$$

and $\tau(q) = 4|q|$. Both transformations date back to [41]. The paths denoted by $y(s)$ correspond to the harmonic oscillator in four dimensions, the ones denoted by $x(t)$ to those of the hydrogen atom. The measures $\mathrm{d}N$ and $\mathrm{d}M$ are the Wiener measures $\mathrm{d}W^4$ and $\mathrm{d}W^3$ on $\mathcal{C}(\mathbb{R}^4, r_a)$ and $\mathcal{C}(\mathbb{R}^3, q_a)$, respectively.

5.2. *Transformation identity*

The content of the following statement is that the transformation identity holds if it holds for the 'free particle'. This condition, called (**C4**), is elementary to check in the sense that no specific knowledge of stochastic calculus is needed. An elementary proof is given in appendix 2.

Statement 2. *Let* $\mathrm{d}N$ *and* $\mathrm{d}M$ *denote measures, associated with Markov processes, on the path spaces* $\mathcal{C}(\mathcal{R}, r_a)$ *and* $\mathcal{C}(\mathcal{Q}, q_a)$*, respectively. Let* K *and* T *be space and time transformations satisfying conditions* (C1), (C2) *and* (C3).

If for arbitrary $r \in \mathcal{R}$, $q' \in \mathcal{Q}$ *and for all* t *the condition*

$$\text{(C4)} \qquad \int_{\mathcal{C}(\mathcal{Q},k(r))} \mathrm{d}M[x]\, \delta(x(t) - q') = \int_{\mathcal{C}(\mathcal{R},r)} \mathrm{d}N[y]\, \delta\Big((TKy)(t) - q'\Big)$$

is fulfilled, then the transformation identity holds (with $q_a = k(r_a)$ *)*

$$\int_{\mathcal{C}(\mathcal{Q},q_a)} \mathrm{d}M[x]\, F[x] = \int_{\mathcal{C}(\mathcal{R},r_a)} \mathrm{d}N[y]\, F[TKy] \tag{33}$$

for a wide class of functionals $F : \mathcal{C}(\mathcal{Q}, q_a) \to \mathbb{R}$.

We stress that the path dependence of the 'pinning time' $s_{Ky}(t)$ on the RHS of (C4) causes no problems, because we consider only normalized measures on open-ended paths. Statement 2 can be considered as the basis of a calculus for path-dependent time transformations in path integration without the need to fall back at each step to discretization prescriptions. It may be used to supply rigour to transformation ideas and their recipe-like application in path-integral calculations.

By setting $V = 0$ in (36) below, the Laplace transform of (C4) is seen to be

$$\text{(C4}'\text{)} \qquad \int_0^\infty \mathrm{d}t\, \mathrm{e}^{Et} m_t(q', k(r)) = \tau(q') \int_0^\infty \mathrm{d}s \int_{\mathcal{C}(\mathcal{R},r)} \mathrm{d}N[y]\, \delta\Big(k(y(s)) - q'\Big) \\ \times \exp\left\{ E \int_0^s \mathrm{d}s'\, \tau\Big(k(y(s'))\Big) \right\}.$$

By the invertibility of the Laplace transformation (C4′) is equivalent to (C4). It is (C4′) which, in applications, is often easier to check.

We remark that within the theory of stochastic differential equations there exist other criteria (replacing (C4)) which lead also to the transformation identity, see e.g. [16–22].

5.3. Calculation of the Green function

In this subsection we are interested in the consequences of the transformation identity (33) for Feynman–Kac functionals (16). As it stands, it is difficult to proceed with (33) as a path-integral representation for the propagator because there occur path-dependent s-times on its RHS. It is only by fixing s-times that one can proceed with the calculation of the path integral. Accordingly, we are going to perform a Laplace transformation of (33) in order to get a path-integral representation for the Green function. To this end we multiply both sides by $\exp\{Et\}$ and integrate over positive t. By (8) and (7), the LHS is immediately identified with the Green function of $\hat{H} = \hat{T}_m + V(\hat{q})$ and (33) takes the form

$$\langle q_b | (\hat{H} - E)^{-1} | q_a \rangle \\ = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{Et} \int_{\mathcal{C}(\mathcal{R},r_a)} \mathrm{d}N[y]\, \delta\Big((Ky)(s_{Ky}(t)) - q_b\Big) \\ \times \exp\left\{ -\int_0^t \mathrm{d}t'\, V\Big((Ky)(s_{Ky}(t'))\Big) \right\}. \tag{34}$$

The important step to make next, is to substitute s for $s_{Ky}(t)$ and s' for $s_{Ky}(t')$ inside the path integral (note that properties **(C2)** and **(C3)** are used). Thus we arrive at

$$\begin{aligned}\langle q_b|(\hat{H}-E)^{-1}|q_a\rangle \\ &= \int_{C(\mathcal{R},r_a)} \mathrm{d}N[y] \int_0^\infty \mathrm{d}s\, \tau\Big((Ky)(s)\Big)\, \mathrm{e}^{Et_{Ky}(s)}\, \delta\Big((Ky)(s)-q_b\Big) \\ &\quad \times \exp\left\{-\int_0^s \mathrm{d}s'\, \tau\Big((Ky)(s')\Big)\, V\Big((Ky)(s')\Big)\right\}. \end{aligned} \tag{35}$$

Summarizing and recalling the remark made below (8) concerning the allowed values E of the energy, we have shown:

Statement 3. *The transformation identity (33) implies the following path-integral representation for the Green function of the Hamiltonian* $\hat{H} = \hat{T}_m + V(\hat{q})$ *(recall the one-to-one correspondences* $\hat{T}_m \to m_t \to \mathrm{d}M$ *and that* $k(r_a) = q_a$ *)*

$$\begin{aligned}\langle q_b|(\hat{H}-E)^{-1}|q_a\rangle \\ &= \tau(q_b) \int_0^\infty \mathrm{d}s \int_{C(\mathcal{R},r_a)} \mathrm{d}N[y]\, \delta\Big(k(y(s))-q_b\Big) \\ &\quad \times \exp\left\{\int_0^s \mathrm{d}s'\, \tau\Big(k(y(s'))\Big)\Big[E - V\Big(k(y(s'))\Big)\Big]\right\}. \end{aligned} \tag{36}$$

In consequence, the representation (36) is valid if the conditions **(C1)**, **(C2)**, **(C3)** and **(C4′)** are fulfilled. This is useful for applications in quantum physics.

The explicit evaluation of the path integral in (36) needs a substitution inside the delta function $\delta(k(y(s))-q_b)$. This requires a closer look for mappings k which are not one-to-one, as is the case, for example, for the hydrogen atom [20].

6. Application: Green function of the one-dimensional Morse system

In this section we apply statements 2 and 3 to calculate the Green function of the Morse system. This is a very illustrative example, because it requires also techniques from sections 3 and 4.

We consider a particle on the Euclidean line $\mathbb{R}$ under the influence of the potential

$$V(q) = V_0\left(\mathrm{e}^{-4\alpha q} - 2\gamma \mathrm{e}^{-2\alpha q}\right) \qquad V_0 > 0 \qquad \alpha,\gamma \in \mathbb{R} \quad \alpha \neq 0. \tag{37}$$

This defines (for $\gamma = 1$) the Morse system [42]. Since the corresponding Hamiltonian is $\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{q})$, the associated Markov process is the one-dimensional Wiener process. Recalling the Feynman–Kac formula (10) we can represent the Green function of the Morse system (for $E < -\gamma^2 V_0$) as

$$\begin{aligned}\langle q_b|(\hat{H}-E)^{-1}|q_a\rangle \\ &= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{Et} \int_{C(\mathbb{R},q_a)} \mathrm{d}W^1[x]\, \delta(x(t)-q_b)\, \exp\left\{-\int_0^t \mathrm{d}t'\, V(x(t'))\right\}. \end{aligned} \tag{38}$$

In order to calculate this non-Gaussian path integral we consider the path space $\mathcal{C}(\mathbb{R}_+, r_a)$ equipped with the measure $\mathrm{d}N = \mathrm{d}B_0$ and perform a suitable space and time transformation to the path space $\mathcal{C}(\mathbb{R}, q_a)$ with the measure $\mathrm{d}M = \mathrm{d}W^1$ of the Morse system

$$TK : \mathcal{C}(\mathbb{R}_+, r_a) \longrightarrow \mathcal{C}(\mathbb{R}, q_a) \qquad y(s) \mapsto x(t) = (TKy)(t). \tag{39}$$

Following [43, 44], we choose

$$q = k(r) = -\alpha^{-1} \ln r \tag{40}$$

and

$$\tau(q) = \alpha^{-2}\, \mathrm{e}^{2\alpha q} \tag{41}$$

hence

$$t_{Ky}(s) = \alpha^{-2} \int_0^s \mathrm{d}s'\, y^{-2}(s'). \tag{42}$$

According to (31) we get the combined transformation

$$x(t) = (TKy)(t) = (Ky)(s_{Ky}(t)) = -\alpha^{-1} \ln y(s_{Ky}(t)). \tag{43}$$

We will show in appendix 3 that these transformations satisfy conditions (C1) to (C3). In addition, we will show there that (C4) holds. Hence, statement 2 leads to the useful transformation identity

$$\int_{\mathcal{C}(\mathbb{R}, q_a)} \mathrm{d}W^1[x]\; F[x] = \int_{\mathcal{C}(\mathbb{R}_+, r_a)} \mathrm{d}B_0[y]\; F[TKy] \tag{44}$$

where F may be a fairly general functional on $\mathcal{C}(\mathbb{R}, q_a)$. It is the central point of this section.

By statement 3 the Green function of the Morse system can be represented as

$$\begin{aligned} \langle q_b | (\hat{H} - E)^{-1} | q_a \rangle &= \frac{1}{|\alpha| r_b} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{\gamma\omega^2 s} \int_{\mathcal{C}(\mathbb{R}_+, r_a)} \mathrm{d}B_0[y]\; \delta(y(s) - r_b) \\ &\quad \times \exp\left\{ -\frac{1}{2} \int_0^s \mathrm{d}s' \left(\omega^2 y^2(s') + \frac{\nu^2}{y^2(s')} \right) \right\} \end{aligned} \tag{45}$$

where we have introduced the non-negative quantities

$$\omega := \sqrt{\frac{2V_0}{\alpha^2}} \qquad \text{and} \qquad \nu := \sqrt{-\frac{2E}{\alpha^2}} \tag{46}$$

($\sqrt{\ }$ denotes the principal branch of the square root) and used the change-of-variable formula for the delta function. Moreover, one should note that $r_a = \mathrm{e}^{-\alpha q_a}$ and $r_b = \mathrm{e}^{-\alpha q_b}$.

This is the point where statement 1 comes in. In order to eliminate the $1/y^2$ term in (45) we proceed as in section 4 by applying statement 1 with $\mu = 0$. This gives

$$\frac{1}{|\alpha| r_b} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{\gamma\omega^2 s} \left(\frac{r_a}{r_b}\right)^\nu \int_{C(\mathbb{R}_+, r_a)} \mathrm{d}B_\nu[y]\ \delta(y(s) - r_b) \times \exp\left\{-\frac{\omega^2}{2}\int_0^s \mathrm{d}s'\, y^2(s')\right\} \tag{47}$$

for the Green function. By (14) and after performing the final s-integration with the help of [35, formula 6.669.4], we end up with

$$\begin{aligned}\langle q_b|(\hat{H} - E)^{-1}|q_a\rangle &= \frac{\Gamma\left(\frac{1}{2}(\nu - \gamma\omega + 1)\right)}{|\alpha|\,\omega\,\Gamma(\nu+1)}\, \mathrm{e}^{\alpha(q_b+q_a)} \\ &\quad \times \left\{\Theta\Big(\alpha(q_a - q_b)\Big)\, W_{\gamma\omega/2,\,\nu/2}\left(\omega \mathrm{e}^{-2\alpha q_b}\right) M_{\gamma\omega/2,\,\nu/2}\left(\omega \mathrm{e}^{-2\alpha q_a}\right) \right. \\ &\quad \left. + (q_a \leftrightarrow q_b)\right\}.\end{aligned} \tag{48}$$

Here Θ denotes Heaviside's unit step function with the convention $\Theta(0) := \frac{1}{2}$, Γ denotes Euler's gamma function, and $M_{\sigma,\rho}$ and $W_{\sigma,\rho}$ denote Whittaker's functions [35, p 1059]. This exact result for the Green function of the Morse system agrees with that in [3], but differs more or less from other results [6, 43]. All these results were obtained by intuitive path-integral calculations.

For completeness and convenience of the reader we present the discrete eigenvalues E_n and corresponding normalized eigenfunctions $\psi_n(q)$ (existing for $\gamma\omega > 1$) of $\hat{H}$ as obtained from the poles and corresponding residues of (48)

$$E_n = -\tfrac{1}{2}(\alpha\nu_n)^2 \qquad \psi_n(q) = |2\alpha|^{1/2}\, \varphi_n(\omega \mathrm{e}^{-2\alpha q}) \tag{49}$$

$$\varphi_n(\xi) := \left(\frac{\nu_n\, \Gamma(n+1)}{\Gamma(\nu_n + n + 1)}\right)^{1/2} \mathrm{e}^{-\xi/2}\, \xi^{\nu_n/2}\, L_n^{(\nu_n)}(\xi) \qquad \xi > 0. \tag{50}$$

Here $L_n^{(\mu)}$ denotes a generalized Laguerre polynomial [35] and n is any non-negative integer such that

$$\nu_n := \gamma\omega - 2n - 1 > 0. \tag{51}$$

These eigenvalues and eigenfunctions generalize results of [45], where the case $\gamma = 1$ was treated by operator methods.

7. Concluding remarks

We have presented two well-founded and practical techniques of how to perform changes of dimension and path-dependent time transformations in path integrals. Thus, rigour is supplied to earlier ideas for the evaluation of certain non-Gaussian path integrals. The power of these techniques has been illustrated by two standard examples of quantum physics. We hope that we have thereby contributed to remove the uneasiness many people have felt about the recipe-like way in which these interesting ideas were applied previously.

We claim that all examples treated within the intuitive approaches are covered by the techniques presented in this paper, possibly with an extension of statement 1 to other families of Markov processes. Here we only mention the Pöschl–Teller and Rosen–Morse systems in addition to the examples mentioned above.

Clearly, it is interesting to see how the techniques described work in the calculation of propagators or Green functions for such one-particle quantum systems. Even so, one must not overestimate the value of path integration for obtaining these types of results, because they can often be obtained more directly with other methods (operator theory, stochastic differential equations). In our opinion, the real efficiency of path integration manifests itself in tackling other problems. For example, path integration may serve to provide new estimates and controlled approximations [12, 13, 47] which are hard to get otherwise as is the case for the celebrated polaron [1, 13, 48]. It would be nice to extend the ideas underlying the above techniques to this and other non-Gaussian and effectively non-Markovian situations.

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Appendix 1. Proof of statement 1 for Feynman–Kac functionals

We assert first that the operator $\hat{T}_\nu := \hat{T}_{b^{(\nu)}} = \frac{1}{2}\{\hat{p}^2 + \mathrm{i}(2\nu+1)\hat{p}\,\hat{q}^{-1}\}$ is the generator of the Bessel process with index ν, see also [46, p 60], that is

$$\langle q'|\exp\{-t\hat{T}_\nu\}|q\rangle = b_t^{(\nu)}(q', q). \tag{52}$$

This can be shown by verifying for (11) the (Fokker–Planck) equation

$$\frac{\partial}{\partial t} b_t^{(\nu)}(q', q) = \frac{1}{2}\left(\frac{\partial^2}{\partial q'^2} - \frac{\partial}{\partial q'}\frac{2\nu+1}{q'}\right) b_t^{(\nu)}(q', q) \tag{53}$$

and observing that the differential operator on the RHS is (except for an overall sign) the action of $\hat{T}_\nu$ in position representation. Note that for $\nu = d/2 - 1$, this is one half of the adjoint of the radial part of the d-dimensional Laplacian. Hence, we have according to (7)

$$\begin{aligned}\int_{C(\mathbb{R}_+,q)} \mathrm{d}B_\nu[x]\ \delta(x(t)-q')\ \exp\left\{-\int_0^t \mathrm{d}t'\ V(x(t'))\right\} \\ &= \langle q'|\exp\left\{-t\left(\hat{T}_\nu + V(\hat{q})\right)\right\}|q\rangle \\ &= \langle q'|\exp\left\{-t\hat{q}^{(\nu-\mu)}\left(\hat{T}_\mu + \frac{\nu^2-\mu^2}{2\hat{q}^2} + V(\hat{q})\right)\hat{q}^{-(\nu-\mu)}\right\}|q\rangle \\ &= \langle q'|\hat{q}^{(\nu-\mu)}\exp\left\{-t\left(\hat{T}_\mu + \frac{\nu^2-\mu^2}{2\hat{q}^2} + V(\hat{q})\right)\right\}\hat{q}^{-(\nu-\mu)}|q\rangle \\ &= \left(\frac{q'}{q}\right)^{\nu-\mu}\int_{C(\mathbb{R}_+,q)} \mathrm{d}B_\mu[x]\ \delta(x(t)-q') \\ &\quad\times\exp\left\{-\int_0^t \mathrm{d}t'\ \left(\frac{\nu^2-\mu^2}{2x^2(t')} + V(x(t'))\right)\right\}.\end{aligned} \tag{54}$$

For the second equality we have used the commutation relation $\hat{q}\hat{p}-\hat{p}\hat{q} = \mathrm{i}$. This completes the proof of statement 1 for Feynman–Kac functionals (16).

Appendix 2. Proof of statement 2

This proof is patterned after the so-called *computational proof* in [20]. It is sufficient to show

$$\begin{aligned}&\int_{C(\mathcal{Q},q_a)} \mathrm{d}M[x]\ \delta(x(t_1)-q_1)\ldots\delta(x(t_n)-q_n) \\ &\quad= \int_{C(\mathcal{R},r_a)} \mathrm{d}N[y]\ \delta\Big((TKy)(t_1)-q_1\Big)\ldots\delta\Big((TKy)(t_n)-q_n\Big)\end{aligned} \tag{55}$$

because this implies (33) for cylindrical functionals f_n and hence for fairly general functionals F. Without loss of generality, we may assume $0 < t_1 < t_2 < \ldots < t_n$. By the Markov property (4) the path integral on the LHS of (55) factorizes into the product (with $q_0 := q_a$ and $t_0 := 0$)

$$\prod_{j=1}^{n}\ \int_{C(\mathcal{Q},q_{j-1})} \mathrm{d}M[x]\ \delta\Big(x(t_j-t_{j-1})-q_j\Big). \tag{56}$$

Using for each factor in (56) condition (C4), equation (55) is seen to be equivalent to the equation (note that $q_j = k(r_j)$)

$$\begin{aligned}&\int_{C(\mathcal{R},r_a)} \mathrm{d}N[y]\ \delta\Big((Ky)(s_{Ky}(t_1))-q_1\Big)\ldots \\ &\qquad\times\int_{C(\mathcal{R},r_{n-1})} \mathrm{d}N[y]\ \delta\Big((Ky)(s_{Ky}(t_n-t_{n-1}))-q_n\Big) \\ &\quad= \int_{C(\mathcal{R},r_a)} \mathrm{d}N[y]\ \delta\Big((Ky)(s_{Ky}(t_1))-q_1\Big)\ldots \\ &\qquad\times\delta\Big((Ky)(s_{Ky}(t_n))-q_n\Big).\end{aligned} \tag{57}$$

We will prove (57)—a consequence of the so-called strong Markov property—by showing that the n-dimensional Laplace transforms of both sides with respect to the difference times $t'_1 := t_1$, $t'_j := t_j - t_{j-1}$, $j = 2, \ldots, n$ are equal. For the transform of the RHS of (57) we obtain

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t'_1 \ \cdots \int_0^\infty \mathrm{d}t'_n \ \exp\left\{\sum_{j=1}^n E_j t'_j\right\} \int_{\mathcal{C}(\mathcal{R},r_a)} \mathrm{d}N[y] \\
&\qquad \times \delta\Big((Ky)(s_{Ky}(t'_1)) - q_1\Big) \ldots \delta\Big((Ky)(s_{Ky}(\textstyle\sum_{l=1}^n t'_l)) - q_n\Big) \\
&\quad = \int_0^\infty \mathrm{d}s'_1 \ \tau(q_1) \ \cdots \int_0^\infty \mathrm{d}s'_n \, \tau(q_n) \int_{\mathcal{C}(\mathcal{R},r_a)} \mathrm{d}N[y] \\
&\qquad \times \exp\left\{ E_1 t_{Ky}(s'_1)\right\} \delta\Big((Ky)(s'_1) - q_1\Big) \ldots \\
&\qquad \times \exp\left\{E_n\Big(t_{Ky}(\textstyle\sum_{l=1}^n s'_l) - t_{Ky}(\sum_{l=1}^{n-1} s'_l)\Big)\right\} \delta\Big((Ky)(\textstyle\sum_{l=1}^n s'_l) - q_n\Big).
\end{aligned} \tag{58}$$

In the last step we have substituted

$$s'_1 := s_{Ky}(t'_1) \qquad s'_j := s_{Ky}(\textstyle\sum_{l=1}^j t'_l) - s'_{j-1} \qquad j = 2, \ldots, n$$

and used (C2) and (C3). The integrand of the last path integral is a product of n functionals, the jth factor of which depending only on the values of the paths between the times $\sum_{l=1}^{j-1} s'_l$ and $\sum_{l=1}^{j} s'_l$. Hence, by the Markov property, this path integral factorizes, giving

$$\begin{aligned}
&\int_0^\infty \mathrm{d}s'_1 \ \tau(q_1) \ \cdots \int_0^\infty \mathrm{d}s'_n \ \tau(q_n) \\
&\qquad \times \int_{\mathcal{C}(\mathcal{R},r_a)} \mathrm{d}N[y] \ \exp\left\{E_1 t_{Ky}(s'_1)\right\} \ \delta\Big((Ky)(s'_1) - q_1\Big) \cdots \\
&\qquad \times \int_{\mathcal{C}(\mathcal{R},r_{n-1})} \mathrm{d}N[y] \ \exp\left\{E_n t_{Ky}(s'_n)\right\} \ \delta\Big((Ky)(s'_n) - q_n\Big).
\end{aligned} \tag{59}$$

But this is exactly what is obtained when taking the n-dimensional Laplace transform of the LHS of (57) with respect to t'_j (as defined above) and substituting $s_j := s_{Ky}(t'_j)$.

Appendix 3. Check of conditions (C1) to (C4) for the Morse system

We want to apply statement 2 in order to prove (44). Thus, we have to check the conditions (C1) to (C4):

Condition on the space transformation (40):

(C1) Clearly, k is a smooth mapping from $\mathbb{R}_+$ onto $\mathbb{R}$.

Conditions on the time transformation (42):

(C2) Obviously, $t_{Ky}(s)$ is strictly increasing in s.

(C3) Observe that

$$t_{Ky}(s) \geqslant \alpha^{-2} \int_0^s \mathrm{d}s'\, \Theta(1 - y(s')). \tag{60}$$

Since by definition and (13) the paths y realize the radial part of a two-dimensional Wiener process, it follows from [12, theorem 7.13] that the RHS of (60) tends to infinity as $s \to \infty$ for almost all y.

Condition of statement 2:

(C4) Equivalently we will show (C4′), that is, the equality

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{Et}\, w_t^{(1)}(-\alpha^{-1}\ln r', -\alpha^{-1}\ln r) = \frac{1}{|\alpha| r'} \int_0^\infty \mathrm{d}s \int_{C(\mathbb{R}_+, r)} \mathrm{d}B_0[y]\, \delta(y(s) - r') \exp\left\{E \int_0^s \frac{\mathrm{d}s'}{\alpha^2 y^2(s')}\right\}. \tag{61}$$

The Laplace transform on the LHS of (61) can be done using [49]. To calculate the RHS we use statement 1 to eliminate the potential term. The resulting 'free-particle' Bessel path integral yields the transition density (11). The final s-integration is performed with the help of [35, formula 6.623.3], thereby completing the proof of (61).

References

[1] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw–Hill)
[2] Duru I H and Kleinert H 1979 *Phys. Lett.* **84B** 185
[3] Kleinert H 1990 *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (Singapore: World Scientific)
[4] Ho R and Inomata A 1982 *Phys. Rev. Lett.* **48** 231
[5] Duru I H and Kleinert H 1982 *Fortschr. Phys.* **30** 401
[6] Pak N K and Sokmen I 1984 *Phys. Rev.* A **30** 1629
[7] Inomata A 1984 *Phys. Lett.* **101A** 253
[8] Steiner F 1986 *Path Integrals from meV to MeV* ed M C Gutzwiller *et al* (Singapore: World Scientific) p 335
[9] Khandekar D C and Lawande S V 1986 *Phys. Rep.* **137** 115
[10] Inomata A 1986 *Path Integrals from meV to MeV* ed M C Gutzwiller *et al* (Singapore: World Scientific) p 433
[11] Kleinert H 1987 *Phys. Lett.* **120A** 361
[12] Simon B 1979 *Functional Integration and Quantum Physics* (New York: Academic)
[13] Roepstorff G 1991 *Pfadintegrale in der Quantenphysik* (Braunschweig: Vieweg)
[14] Inomata A 1989 *Path Integrals from meV to MeV* ed V Sa-yakanit *et al* (Singapore: World Scientific) p 112
[15] Junker G 1990 *J. Phys. A: Math. Gen.* **23** L881
[16] Blanchard Ph and Sirugue M 1981 *J. Math. Phys.* **22** 1372

[17] Blanchard Ph 1982 *Stochastic Processes in Quantum Theory and Statistical Physics (Lecture Notes in Physics 173)* ed S Albeverio *et al* (Berlin: Springer) p 19

[18] Young A and DeWitt-Morette C 1986 *Ann. Phys., NY* **169** 140

[19] Storchak S N 1989 *Phys. Lett.* **135A** 77

[20] Castrigiano D P L and Stärk F 1989 *J. Math. Phys.* **30** 2785

[21] Castrigiano D P L and Stärk F 1990 *White Noise Analysis, Mathematics and Applications* ed T Hida *et al* (Singapore: World Scientific) p 49

[22] Castrigiano D P L and Stärk F 1991 *Quantum Probability VI* ed L Accardi (Singapore: World Scientific) p 245

[23] Revuz D and Yor M 1991 *Brownian Motion and Continuous Martingales* (Berlin: Springer)

[24] Edwards S F and Gulyaev Y V 1964 *Proc. Roy. Soc.* A **279** 229

[25] Peak D and Inomata A 1969 *J. Math. Phys.* **10** 1422

[26] McLaughlin D W and Schulman L S 1971 *J. Math. Phys.* **12** 2520

[27] Khandekar D C and Lawande S V 1975 *J. Math. Phys.* **16** 384

[28] Langguth W and Inomata A 1979 *J. Math. Phys.* **20** 499

[29] Pitman J and Yor M 1981 *Stochastic Integrals (Lecture Notes in Mathematics 851)* ed D Williams (Berlin: Springer) p 285

[30] Kolmogoroff A 1933 (reprint 1977) *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Berlin: Springer)

[31] Feller W 1971 *An Introduction to Probability Theory and its Applications* vol 2 (New York: Wiley)

[32] Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)

[33] Karatzas I and Shreve S E 1988 *Brownian Motion and Stochastic Calculus* (New York: Springer)

[34] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics* vol I *Functional Analysis* (New York: Academic)

[35] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)

[36] Erdélyi A *et al* 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill) formulae 7.8 (11) and 11.1.1 (11)

[37] Bratteli O and Robinson D W 1981 *Operator Algebras and Quantum Statistical Mechanics* vol 2 (Berlin: Springer)

[38] Landau L D and Lifshitz E M 1977 *Course of Theoretical Physics* vol 3 *Quantum Mechanics* (Oxford: Pergamon) section 35

[39] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* vol II *Fourier Analysis, Self-Adjointness* (New York: Academic)

[40] Calogero F 1969 *J. Math. Phys.* **10** 2191

[41] Kustaanheimo P and Stiefel E 1965 *J. Reine Angew. Math.* **218** 207

[42] Morse P M 1929 *Phys. Rev.* **34** 57

[43] Cai P J, Inomata A and Wilson R 1983 *Phys. Lett.* **96A** 117

[44] Duru I H 1983 *Phys. Rev.* D **28** 2689

[45] Nieto M M and Simmons, Jr L M 1979 *Phys. Rev.* D **20** 1342

[46] Itô K and McKean H P 1974 *Diffusion Processes and their Sample Paths* (Berlin: Springer)

[47] Leschke H 1988 *Path Summation: Achievements and Goals* ed S Lundqvist *et al* (Singapore: World Scientific) p 399

[48] Adamowski J, Gerlach B and Leschke H 1980 *Functional Integration—Theory and Applications* ed J-P Antoine and E Tirapegni (New York: Plenum) p 291

[49] Roberts G E and Kaufman H 1966 *Table of Laplace Transforms* (London: Saunders) formula 3.2.6